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Estimate for regeneration up to the golden rule time

I Antoniou†, J Levitan†§ and L P Horwitz‡‖

† Service de Chimie Physique, Université Libre de Bruxelles, CP 231, Campus Plaine ULB, B-1050 Brussels, Belgium

‡ Institute for Advanced Study, School of Natural Sciences Princeton, NJ 08540, USA

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Abstract. We study the regeneration contribution to the decay law in Wigner–Weisskopf theory for times less than and up to the golden rule time. A power series expansion for the regeneration term and the part of the product of the amplitudes which has the semigroup property is carried out in second-order perturbation theory, the same order to which the Wigner–Weisskopf calculation is carried out in their estimate of the line widths in atomic decay. We show that the regeneration contribution has a smaller leading behaviour in t than the amplitudes at times of the order of the golden rule time, thus accounting for an *approximate* semigroup behaviour, on this scale, within the framework of the Wigner–Weisskopf theory. For very short times, the estimates of Misra and Sinha are obtained.

In their historic paper [1] of 1930, Wigner and Weisskopf showed that the decay width of an unstable system appears linearly in the decay law in second-order perturbation theory, in the range of times for which the golden rule becomes valid. At the golden rule time, the $O(t^2)$ initial mode of unstable system decay disappears, and the system may enter what appears to be an exponentially dominated mode (the possible effects of this early time development have been studied quite extensively [2]). The dynamics of the onset of approximate exponential decay, signalled by the appearance of the decay width in the work of Wigner and Weisskopf [1], should therefore be visible in the framework of second-order perturbation theory.

In the method of Wigner and Weisskopf [1], the amplitude for the system to be found in the initial state ϕ is

$$A(t) = (\phi, e^{-iHt}\phi) \tag{1}$$

where H is the total Hamiltonian of the system. If the evolution were semigroup, then $A(t)$ would satisfy the relation $A(t_1)A(t_2) = A(t_1 + t_2)$. An amplitude of the form (1) cannot, however, have this property exactly. We define the correction term as follows. Assuming that the Hilbert space is spanned by the discrete state ϕ (the unstable state) and a continuum $\{|\lambda\rangle\}$, $A(t_1 + t_2)$ is given by

$$\begin{aligned} A(t_1 + t_2) &= (\phi, e^{-iHt_1}e^{-iHt_2}\phi) \\ &= A(t_1)A(t_2) + R(t_1, t_2) \end{aligned} \tag{2}$$

§ Permanent address: College of Judea and Samaria, Ariel, and Department of Physics, Bar Ilan University, Ramat Gan, Israel.

‖ Permanent address: School of Physics, Raymond and Beverly Sackler Faculty of Exact Sciences, Tel Aviv University, Ramat Aviv, Israel; also at Department of Physics, Bar Ilan University, Ramat Gan, Israel.

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where the second term on the right-hand side

$$R(t_1, t_2) = \int d\lambda\, (\phi|e^{-iHt_1}|\lambda)(\lambda|e^{-iHt_2}|\phi) \tag{3}$$

is called the *regeneration term*. This term does not depend on $t_1 + t_2$, but on the times t_1, t_2 separately. It corresponds to the non-Markovian part of the decay process; as seen from the formula (3), it contains the integral, over all λ, of the product of the amplitude for transition from ϕ to $|\lambda)$ (decay) over an interval t_2 with the amplitude for the inverse transition (regeneration), to ϕ during the interval t_1.

To the extent that in the period of time for which the exponential behaviour dominates, $A(t)$ has an approximate semigroup property, the regeneration term should be small. Our purpose here is to study the behaviour of the regeneration term relative to that of the semigroup part $A(t_1 + t_2)$ of the product of the amplitudes. Since the onset of exponential decay indicates the development of an approximate semigroup behaviour, the regeneration term, which measures the deviation from semigroup behaviour, should diminish relative to the semigroup term.

Even though the Wigner–Weisskopf mathematical description of the process, which we study, is based on a reversible description, the mechanism for the diminishing relative size of the regeneration term should shed some light on the structure of the fundamental processes underlying exponential decay.

As we have remarked above, the decay width emerges linearly in the Wigner–Weisskopf calculation in second-order perturbation theory. This corresponds to the first non-trivial term of the Taylor expansion of the exponential decay law. For the times up to which this calculation is a good approximation (as we shall see, essentially the golden rule time), this second-order approximation permits an examination of the decay law. In the same way, and with the same accuracy, second-order perturbation theory should permit the investigation of the regeneration effect. In fact, since we are interested in an essentially qualitative property of unstable systems, in any actual computation, we can take the coupling constant (the scale of V) to be as small as we wish, and hence higher-order terms can be made arbitrarily small in the neighbourhood of the golden rule time. We therefore compute here the amplitude $A(t)$ and the regeneration term in second-order perturbation theory. For a Hamiltonian of the form

$$H = H_0 + V \tag{4}$$

where V is the perturbation inducing decay of the state ϕ, we may expand the solution of the interaction picture equation

$$i\frac{\partial}{\partial t}\phi_t = V(t)\phi_t \tag{5}$$

where $V(t) = e^{iH_0t}Ve^{-iH_0t}$, as

$$\phi_t \cong \phi - i\int_0^t V_{t'}|\phi)\, dt' + (-i)^2 \int_0^t dt' \int_0^{t'} dt''\, V_{t'}V_{t''}|\phi). \tag{6}$$

Further, assuming $(\phi|V|\phi) = 0$ (this contribution has only the effect of an energy shift), we obtain

$$\begin{aligned} e^{imt}A(t) &= e^{imt}(\phi, \phi_t) \\ &= 1 - \int_0^t dt' \int_0^{t'} dt'' \int_0^{\infty} d\lambda\, W(\lambda)e^{-i(\lambda-m)(t'-t'')} + \cdots \end{aligned} \tag{7}$$

where

$$W(\lambda) = |\langle\lambda|V|\phi\rangle|^2$$

and m is the discrete eigenvalue of H_0 associated with the state ϕ. Retaining only the second-order term, we see that

$$e^{imt}A(t) \cong 1 + \int d\lambda \frac{W(\lambda)}{(\lambda - m)^2}\{e^{-i(\lambda-m)t} + i(\lambda - m)t - 1\}. \tag{8}$$

We remark that an estimate for the golden rule time can be obtained directly from (8). The 'survival' probability to this order is

$$|A(t)|^2 \cong 1 - 4\int d\lambda \frac{W(\lambda)}{(\lambda - m)^2}\frac{\sin^2 \frac{1}{2}(\lambda - m)t}{(\lambda - m)^2}. \tag{9}$$

The limit of large t in which this expression is linear in t, i.e. the application of

$$\frac{\sin^2 xt}{x^2} \sim \pi t \delta(x) \tag{10}$$

in which the trigonometric function with its quadratic denominator can be taken as an effective δ-function times t, is determined by the rate of change of the function $W(\lambda)$, which acts as a test function for the distribution. We know from (9) that for large t the contribution to the integral is from λ in some small neighbourhood of m. In fact the golden rule time is estimated through the relation

$$\lambda - m \leqslant 2\pi/t. \tag{11}$$

We wish to estimate $\lambda - m$ in terms of the variation of $W(\lambda)$. We therefore expand the function $W(\lambda)$ in the neighbourhood of small $\lambda - m$

$$W(\lambda) \cong W(m) + (\lambda - m)W'(m). \tag{12}$$

The correction term is small for

$$\lambda - m \ll \frac{W(m)}{W'(m)}$$

or, from (11), that

$$t \gg 2\pi\frac{W'(m)}{W(m)} = 2\pi\frac{d}{d\lambda}\ln\ W(\lambda)|_{\lambda=m} \tag{13}$$

providing an estimate of the golden rule time in terms of the variation of $W(\lambda)$. For [3] $W(\lambda) \propto \lambda e^{-\mu\lambda}$, one obtains, e.g. for $\mu = 1/2m$,

$$t_{GR} \cong 2\pi/m. \tag{14}$$

As another example

$$W(\lambda) \propto \frac{1}{(\lambda^2 + a^2)}e^{-\mu/\lambda}$$

leads to the same estimate. If $W'(m)$ vanishes, instead of (12), we have

$$W(\lambda) \cong W(m) + \tfrac{1}{2}(\lambda - m)^2 W''(m) \tag{15}$$

and then

$$t_{\mathrm{GR}} \cong 2\pi \sqrt{\frac{W''(m)}{W(m)}}. \tag{16}$$

For $W(\lambda) \propto \lambda \mathrm{e}^{-\mu\lambda}$, with $\mu = 1/m$, this case occurs, but then (16) provides the estimate $t_{\mathrm{GR}} \cong 2\pi/m$, which is of the same order.

We now turn to a parallel second-order expansion of the regeneration term. We obtain an expression for this by computing the transition element $\langle\lambda|\mathrm{e}^{-\mathrm{i}Ht}|\phi\rangle$ to first order, i.e.,

$$\begin{aligned}\langle\lambda|\mathrm{e}^{-\mathrm{i}Ht}|\phi\rangle = \mathrm{e}^{-\mathrm{i}\lambda t}\langle\lambda|\phi_t\rangle &\cong -\mathrm{i}\, \mathrm{e}^{-\mathrm{i}\lambda t}\int_0^t \langle\lambda|V_{t'}|\phi\rangle\,\mathrm{d}t' \\ &\cong \frac{\mathrm{e}^{-\mathrm{i}mt} - \mathrm{e}^{-\mathrm{i}\lambda t}}{m-\lambda}\langle\lambda|V|\phi\rangle.\end{aligned} \tag{17}$$

Then, from (3), to second order†

$$R(t,t') = \mathrm{e}^{\mathrm{i}m(t-t')}\int \mathrm{d}\lambda\,\{1 + \mathrm{e}^{\mathrm{i}(\lambda-m)(t-t')} - \mathrm{e}^{\mathrm{i}(\lambda-m)t} - \mathrm{e}^{-\mathrm{i}(\lambda-m)t'}\}\frac{W(\lambda)}{(m-\lambda)^2}. \tag{18}$$

An upper bound is obtained by replacing the integrand by its absolute value, i.e.

$$|R(t,t')| \leqslant 2\int \mathrm{d}\lambda\,[(1-\cos(\lambda-m)t)(1-\cos(\lambda-m)t')]^{1/2}\frac{W(\lambda)}{(m-\lambda)^2} \tag{19}$$

or, equivalently

$$|R(t,t')| \leqslant 4\int \mathrm{d}\lambda\,\sin\frac{(\lambda-m)t}{2}\sin\frac{(\lambda-m)t'}{2}\frac{W(\lambda)}{(\lambda-m)^2}. \tag{20}$$

Note that for the transition $t' \to t' + t$, for $t' \to t_{\mathrm{GR}}$, the bound (20) is restricted by

$$\frac{\sin\frac{1}{2}(\lambda-m)t'}{\frac{1}{2}(\lambda-m)} \sim \pi\delta(\lambda-m)$$

and hence

$$|R(t,t')|_{t'\sim t_{\mathrm{GR}}} \leqslant t\pi W(m). \tag{21}$$

On the other hand, for t, t' small, we obtain the bound

$$|R(t,t')| \leqslant tt'\int_0^\infty W(\lambda)\,\mathrm{d}\lambda \tag{22}$$

† Note that, from (9) and (18), $|R(t,t)| = 1 - |A(t)|^2$, hence if $A(t)$ decreases, $|R(t,t)|$ must *increase*.

where the last estimate assumes $W(\lambda)$ decreasing sufficiently rapidly so that $\sin(\lambda-m)t/2 \sim (\lambda - m)t/2$ is a good approximation for small t. Note that $\int_0^\infty W(\lambda)\,d\lambda = (\Delta H)^2$ is the 'dispersion' of the full Hamiltonian in the state ϕ.

The bound (22) is in agreement with the results of Misra and Sinha [4], where it was proved that if the small t, t' behaviour is of the order of $t^\alpha t'^\alpha$ for $\alpha > 1$, then the semigroup property must hold exactly (and hence the regeneration term must vanish).

We may obtain an estimate for larger times by expanding $A(t+t')$ and $R(t, t')$ in power series. The coefficients of powers of t, t' are the same set of numbers in each case, and we may therefore compare the two series directly. Consider the expansions of the second-order perturbation theory results of (8) and (18)

$$e^{im(t+t')}A(t+t') = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k)!}\left(a_k - i\frac{b_k(t+t')}{2k+1}\right)(t+t')^{2k} \tag{23}$$

$$R(t,t') = e^{im(t-t')}\sum_{k=1}^{\infty}\frac{(-1)^k}{(2k)!}\left\{a_k[(t-t')^{2k} - t^{2k} - t'^{2k}] + \frac{ib_k}{2k+1}[(t-t')^{2k+1} - t^{2k+1} + t'^{2k+1}]\right\} \tag{24}$$

where

$$a_k = \int d\lambda\, W(\lambda)(\lambda - m)^{2k-2} \geqslant 0 \qquad b_k = \int d\lambda\, W(\lambda)(\lambda - m)^{2k-1}. \tag{25}$$

We see that the leading powers of t, t' in the semigroup part of the product $A(t)A(t')$ precisely cancel in both the real and imaginary parts of $R(t, t')$. This mechanism causes a dominance of the semigroup property, i.e. a tendency towards the relative decrease of the regeneration contributions in the neighbourhood of the golden rule time where the leading powers in t, t' can be expected to dominate the contribution of each of the coefficients a_k, b_k for each fixed t' (or t). We estimate this effect for small t' as follows. As we have pointed out above, one finds, in many interesting cases, $t_{GR} \sim 1/m$. The condition for dominance of the leading term in t for fixed t' in the (rapidly convergent) expansions (23), (24) is, for each k in the series, that

$$t' \ll t/k. \tag{26}$$

Since the regeneration series starts with the next to leading term in powers of t, the condition for the dominance of the varying part of the semigroup over the regeneration term coincides with (26), i.e. for each k, the comparable terms in the expansion are in the ratio kt'/t. We note from (25) that a_k, b_k do not grow quickly in k if we assume that $W(\lambda)$ falls off quickly above $\lambda \sim m$; they are then approximately bounded by $m^k \int d\lambda\, W(\lambda)$. For example, for $k \leqslant 10$, the inequality (26) is satisfied for $t' \ll \frac{1}{10}t_{GR}$. Taking the $k = 1$ contribution, we have the estimate

$$\frac{R(t,t')}{e^{im(t+t')}A(t+t') - 1} \approx t'/t. \tag{27}$$

For t, t' both small, this bound is consistent with (22), (23), but we see that the bound remains valid for t up to the order of the golden rule time. We furthermore see that (in the shift from t' to $t + t'$) the initial regeneration rate at $t = 0$ is given, from (24), by

$$\frac{d}{dt}e^{-im(t-t')}R(t,t')|_{t=0} = \sum_{k=1}^{\infty}\frac{(-1)^k}{(2k)!}\{-2kt'^{2k-1}a_k + ib_k t'^{2k}\}$$

which, for small t' is

$$\frac{d}{dt}e^{-im(t-t')}R(t,t')|_{t=0} = t'a_1 + O(t'^2). \tag{28}$$

Hence, the initial regeneration rate for small t' reaches the bound (22), and, as pointed out in [4, 5], cannot be arbitrarily small.

We have shown that there is a mechanism, in the framework of second-order perturbation theory, for the emergence of the semigroup law of decaying systems in the neighbourhood of times where the decay law enters, according to Wigner–Weisskopf theory, the linear onset of the exponentially dominated mode. It is actually a dynamical question, depending on the spectral function $W(\lambda)$ and the resulting coefficients $\{a_k, b_k\}$, whether the exponential mode emerges as dominant at the golden rule time. We have seen, however, that the leading powers of the regeneration term suffer a cancellation that can cause dominance of the approximate semigroup property. This bound has been demonstrated for (non-vanishing) values of t' small compared to the golden rule time t_{GR}, and for t of the order of t_{GR}. Our bound agrees with results of Misra and Sinha [4] for small t, t', but extends the analysis up to the order of the onset of the approximate exponential decay. A further study of the physical origin of this cancellation would be of interest in obtaining more insight into the mechanism of the decay process.

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